電磁場是向量還是張量？

　電磁場到底是向量還是張量？答案可說「以上皆是」，亦可說是「以上皆非」。聽來雖然荒謬，但我們能否從中得到什麼啟示(如「假向量」和「反夸克」的本質)？

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最簡單的問題vs.最簡單的答案

- 當然是張量！

錯誤的問題vs.錯誤的答案

- 錯誤的答案
- 錯誤的問題

該怎麼問才算正確？

- 什麼是張量？
- 方陣＝二階張量？
- M_{ij}: 二階張量的分量？
- 1xn矩陣＝向量（一階張量）？
- V_i: 向量的分量？

張量必須用「變換式」定義

$$\mathbb{R}^n : x\text{−system} \Rightarrow y\text{-system}$$

$$V^a(y) = \frac{\partial y^a}{\partial x^m} V^m(x), \text{contravariant}$$ 逆變(反變)

$$V_a(y) = \frac{\partial x^m}{\partial y^a} V_m(x), \text{covariant}$$ 協變(共變)

張量必須用「變換式」定義(續)

$$T^{ab}(y) = \frac{\partial y^a}{\partial x^m}\frac{\partial y^b}{\partial x^n}T^{mn}(x), \ \binom{2}{0} \text{ type}$$

$$T_{ab}(y) = \frac{\partial x^m}{\partial y^a}\frac{\partial x^n}{\partial y^b}T_{mn}(x), \ \binom{0}{2} \text{ type}$$

$$T^a_b(y) = \frac{\partial y^a}{\partial x^n}\frac{\partial x^n}{\partial y^b}T^m_n(x), \ \binom{1}{1} \text{ type.}$$

Linear Transformation

$$y^a = R^a_{.m} x^m; \;\; x^m = (R^{-1})^m_{.a} y^a$$

$$V^a(y) = R^a_{.m} V^m(x)$$

$$\boxed{V' = RV}$$

$$V_a(y) = (R^{-1})^m_{.a} V_m(x) = (\tilde{R}^{-1})_a^{.m} V_m(x)$$

Linear Transformation (Cont.)

$$T^{ab}(y) = R^a_{.m} R^b_{.n} T^{mn}(x)$$

$$T_{ab}(y) = (\tilde{R}^{-1})_a^{.m} (\tilde{R}^{-1})_b^{.n} T_{mn}(x)$$

$$T^a_b(y) = R^a_{.m} (\tilde{R}^{-1})_b^{.n} T^m_n(x)$$

Electromagnetic Tensor in Minkowski Space $\mathbb{R}^{3,1}$

$$F_{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{ab} = \begin{pmatrix} 0 & -\tilde{\mathbf{E}} \\ \mathbf{E} & \mathbf{M} \end{pmatrix}$$

Electromagnetic Tensor

- Rank-2 tensor under SO(3,1), i.e.
  Rank-2 tensor under Lorentz transformation.
- E_i, B_j are components of the rank-2 tensor.

$$F^{ab}(y) = L^a_{.m} L^b_{.n} F^{mn}(x)$$

$$F_{ab}(y) = (\tilde{L}^{-1})_a^{.m} (\tilde{L}^{-1})_b^{.n} F_{mn}(x)$$

$$\boxed{\begin{aligned} \partial_a F^{ab} &= 4\pi J^b \\ \partial_{[a} F_{bc]} &= 0 \end{aligned}}$$

$$F^{ab}(y) = L^{a}_{.m} L^{b}_{.n} F^{mn}(x)$$

$$\Longleftrightarrow \quad F' = L F \tilde{L}$$

$$F^{ab}(y) = L^{a}_{.m} L^{b}_{.n} F^{mn}(x)$$

$$= L^{a}_{.m} F^{mn}(x) L^{b}_{.n}$$

$$= L^{a}_{.m} F^{mn}(x) \tilde{L}_{n}^{.b}$$

$$= [L F(x) \tilde{L}]^{ab}$$

Special case: SO(3,1)➔SO(3)

$$L(SO(3)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{pmatrix}$$

$$LF\tilde{L} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{pmatrix} \begin{pmatrix} 0 & -\tilde{\mathbf{E}} \\ \mathbf{E} & \mathbf{M} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{R} \end{pmatrix}$$

$$L F \tilde{L} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{pmatrix} \begin{pmatrix} 0 & -\tilde{\mathbf{E}} \\ \mathbf{E} & \mathbf{M} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{R} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\tilde{\mathbf{E}} \\ R\mathbf{E} & R\mathbf{M} \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{R} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\tilde{\mathbf{E}}\tilde{R} \\ R\mathbf{E} & R\mathbf{M}\tilde{R} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\tilde{\mathbf{E}}' \\ \mathbf{E}' & \mathbf{M}' \end{pmatrix}$$

$$\boxed{\begin{aligned} \mathbf{E}' &= R\mathbf{E} \\ \mathbf{M}' &= R\mathbf{M}\tilde{R} \end{aligned}}$$

Under SO(3)

$$\mathbf{E}' = R\mathbf{E}$$

Vector

$$\mathbf{M}' = R\mathbf{M}\tilde{R}$$

Rank-2 Tensor

$$\mathbf{M}^{ij} = \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix}$$

$$\mathbf{M}^{ij} = \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix}$$

$$\mathbf{M}^{ij} = -\epsilon^{ijk} B_k, \quad B_i = -\frac{1}{2}\epsilon_{ijk}\mathbf{M}^{jk}$$

Tensor contraction?

Is ϵ_{ijk} a rank-3 tensor under $SO(3)$?

Claim1: ϵ_{ij} is a rank-2 tensor under $SO(2)$

$$\det R = R^1_{.1}R^2_{.2} - R^1_{.2}R^2_{.1} = \epsilon^{mn}R^1_{.m}R^2_{.n} = \frac{1}{2!}\epsilon_{ij}\epsilon^{mn}R^i_{.m}R^j_{.n} = 1$$

$$\boxed{\epsilon^{mn}\epsilon_{pq} = \delta^m_p\delta^n_q - \delta^m_q\delta^n_p}$$

$$\frac{1}{2!}\epsilon_{ij}\epsilon^{mn}\epsilon_{pq}R^i_{.m}R^j_{.n} = \epsilon_{pq}$$

$$\frac{1}{2!}\epsilon_{ij}(\delta^m_p\delta^n_q - \delta^m_q\delta^n_p)R^i_{.m}R^j_{.n} = \epsilon_{pq}$$

$$\frac{1}{2!}\epsilon_{ij}(R^i_{.p}R^j_{.q} - R^i_{.q}R^j_{.p}) = \epsilon_{pq}$$

$$\boxed{R^i_{.p}R^j_{.q}\epsilon_{ij} = \epsilon_{pq}}$$

Claim2: ϵ_{ijk} is a rank-3 tensor under $SO(3)$

$$\det R = \boxed{\frac{1}{3!}\epsilon_{ijk}\epsilon^{mn\ell}R^{i}_{.m}R^{j}_{.n}R^{k}_{.\ell} = 1}$$

$$R^{i}_{.p}R^{j}_{.q}R^{k}_{.r}\epsilon_{ijk} = \epsilon_{pqr}$$

Constant Tensor

$$\boxed{R^{i}_{.m}R^{j}_{.m}R^{k}_{.\ell}\epsilon_{ijk} = \epsilon_{mn\ell}}$$

Conclusion:
Under SO(3)

$$1. B_i \in \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix}$$

is a component of a rank-2 tensor.

$$\boxed{\mathbf{M}^{ij} = -\epsilon^{ijk} B_k, \;\; B_i = -\tfrac{1}{2}\epsilon_{ijk}\mathbf{M}^{jk}}$$

$2. B_i \in (B_1,\, B_2,\, B_3)$ is a component of a vector.

Case study1. **B** is a pseudo-vector

A pseudo-vector (or axial vector) is a quantity that transforms like a vector under a proper rotation, but in three dimensions gains an additional sign flip under an improper rotation such as a reflection.

http://en.wikipedia.org/wiki/Pseudovector

$$V' = (\det R) R V = \pm R V$$

$$SO(3) \Longrightarrow O(3): \;\; R \in O(3), \, \det R = \pm 1$$

For improper rotation, $\det R = -1$,

$$\det R = \tfrac{1}{3!} \epsilon_{ijk} \epsilon^{mn\ell} R^{i}_{.m} R^{j}_{.n} R^{k}_{.\ell} = -1$$

$$\epsilon_{ijk} R^{i}_{.m} R^{j}_{.n} = -\epsilon_{mn\ell} (R^{-1})^{\ell}_{.k} = -\epsilon_{mn\ell} R_{k}^{.\ell}$$

$$\mathbf{M}^{ij} = -\epsilon^{ijk} B_k, \quad B_i = -\tfrac{1}{2}\epsilon_{ijk}\mathbf{M}^{jk}$$

ϵ_{ijk} is invariant under transformation.

$$B'_i = -\tfrac{1}{2}\epsilon_{ijk}\mathbf{M}'^{jk}$$

$$= -\tfrac{1}{2}\epsilon_{ijk}R^j_{.m}R^k_{.n}\mathbf{M}^{mn}$$

$$\epsilon_{ijk}R^i_{.m}R^j_{.n} = -\epsilon_{mn\ell}(R^{-1})^{\ell}_{.k} = -\epsilon_{mn\ell}R_k^{:\ell}$$

$$= -\tfrac{1}{2}(-\epsilon_{mn\ell}R_i^{:\ell})\mathbf{M}^{mn}$$

$$= -R_i^{:\ell}[-\tfrac{1}{2}\epsilon_{mn\ell}\mathbf{M}^{mn}]$$

$$= -R_i^{:\ell}B_\ell$$

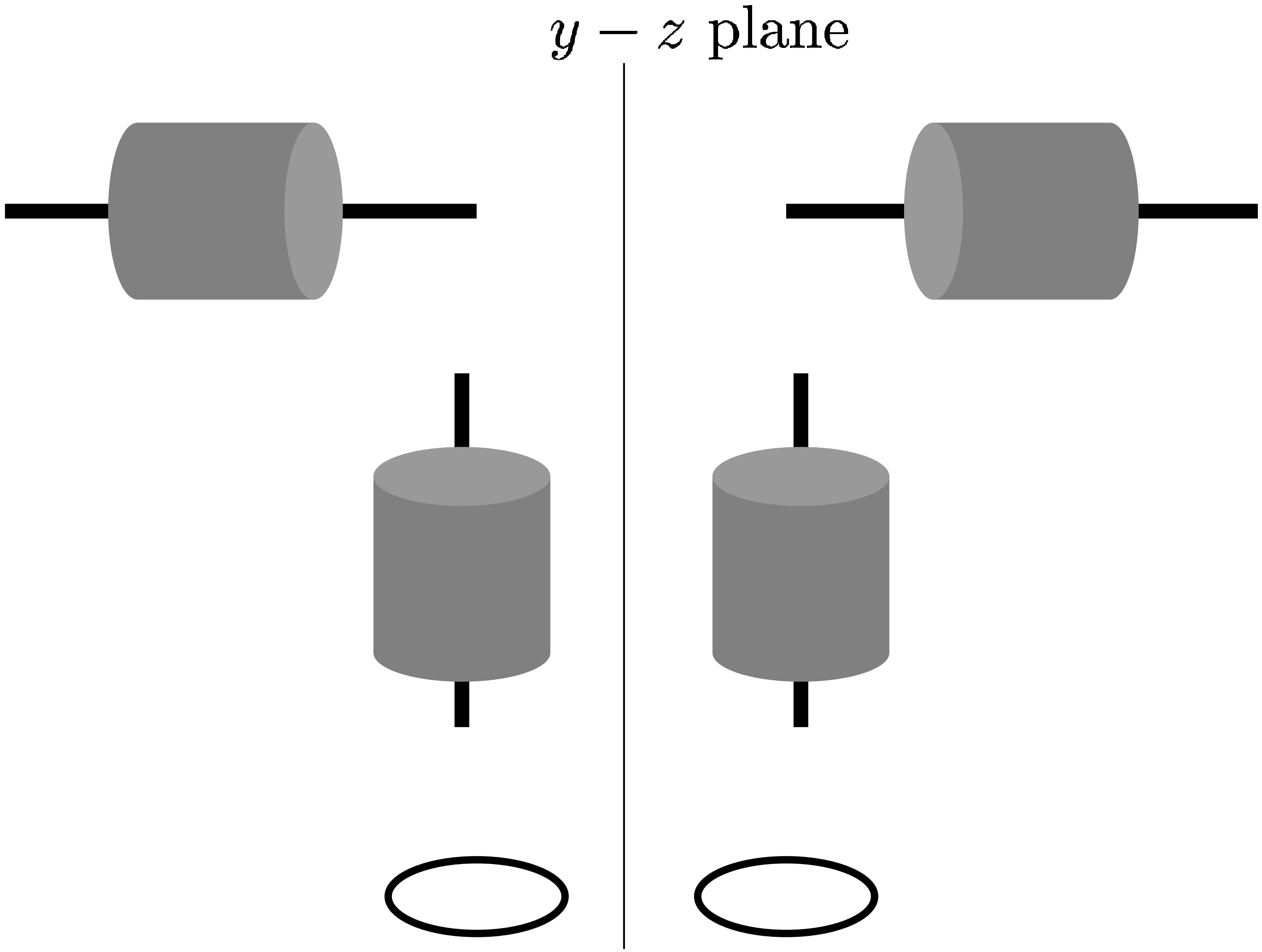
x−reflection $\quad \mathbf{M}^{ij} = \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix}, \quad \mathbf{M}' = R\mathbf{M}\tilde{R}$

$$\mathbf{M}'^{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & -B_x \\ B_y & B_x & 0 \end{pmatrix}$$

$$\Leftrightarrow \quad \mathbf{B}' = -\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{B}$$

Ampère's right hand rule ➔left hand rule?



Case study 2.

1.Cross product of two vectors $A \times B$
is still a vector.

2.Cross product of two vectors $A \times B$
is a pseudovector.

2'.Cross product of two polar vectors $A \times B$
is a axial vector.

Case study 3.
What's special for anti-symmetry

T^{ab} is a general rank-2 tensor in $\mathbb{R}^3$.

$$T^{ab} = S^{ab} + A^{ab}$$

$$\frac{1}{2!}\epsilon_{abc}T^{ab} = \frac{1}{2!}\epsilon_{abc}(S^{ab} + A^{ab}) = \frac{1}{2!}\epsilon_{abc}A^{ab}$$

Rearranging the component

Case study 4.
Generalization by an example

T^{abc} is a rank-3 tensor under $SO(5)$ in $\mathbb{R}^5$,

Then $\frac{1}{3!}\epsilon_{abcde}T^{abc}$ is a rank-2 tensor.

$$\frac{1}{3!}\epsilon_{abcde}T^{abc} = \frac{1}{3!}\epsilon_{abcde}(S^{abc} + M^{abc} + A^{abc}) = \frac{1}{3!}\epsilon_{abcde}A^{abc}$$

Case study 5. Anti-quarks

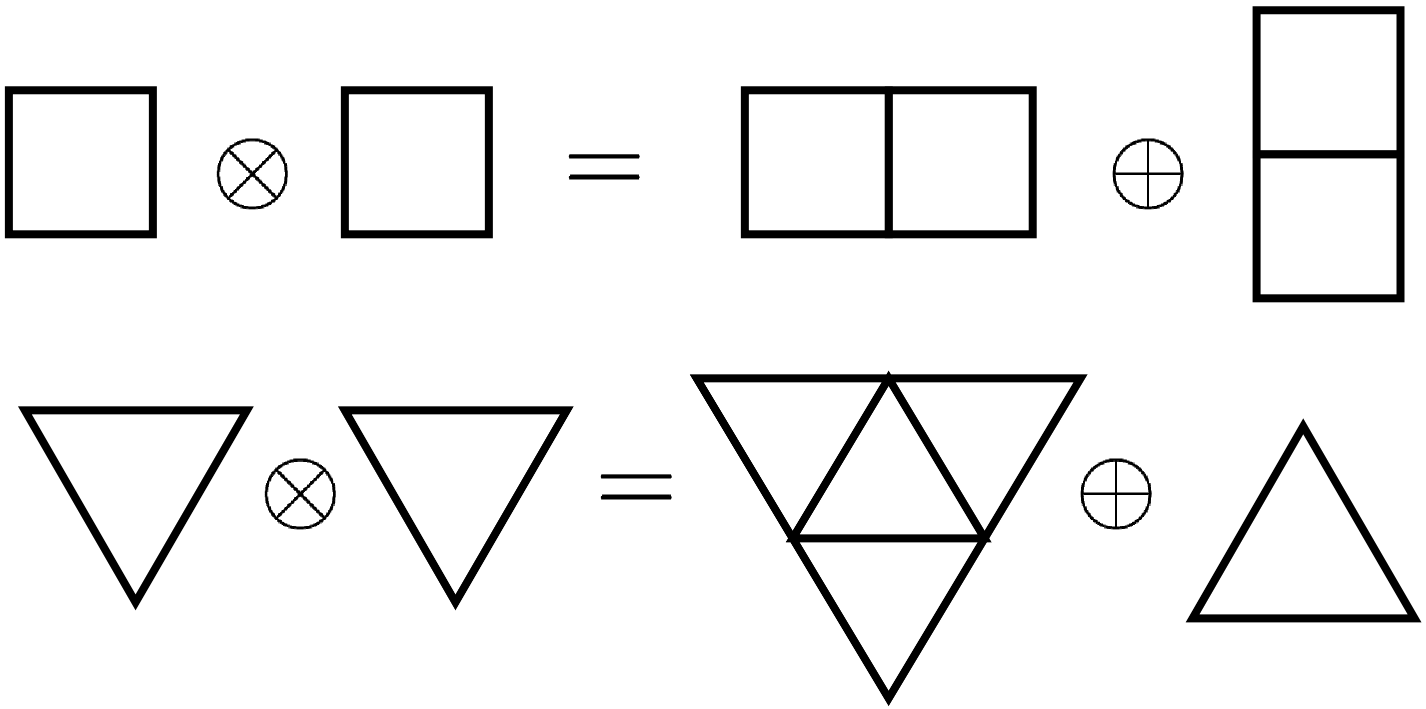
Quarks $\in \binom{1}{0}$ type tensor under $SU(3)$.

$\binom{2}{0}$ type anti-symmetric tensor $\equiv \binom{0}{1}$type tensor.

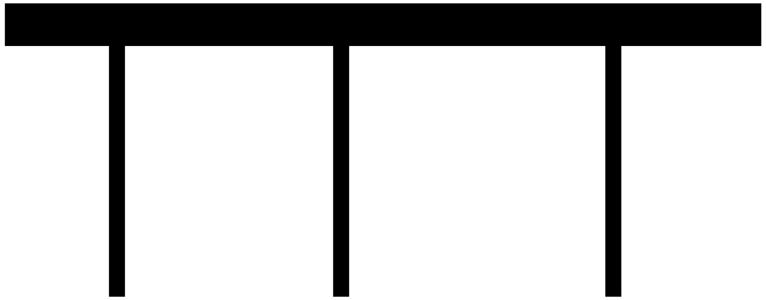
$$ud - du \Rightarrow \bar{s}$$
$$ds - sd \Rightarrow \bar{u}$$
$$su - us \Rightarrow \bar{d}$$

SU(3)

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$$



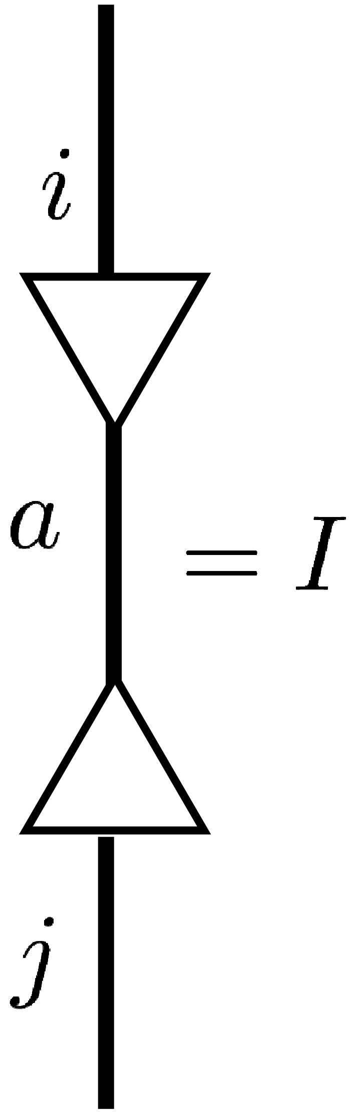
Case study 6. Graphical rep.

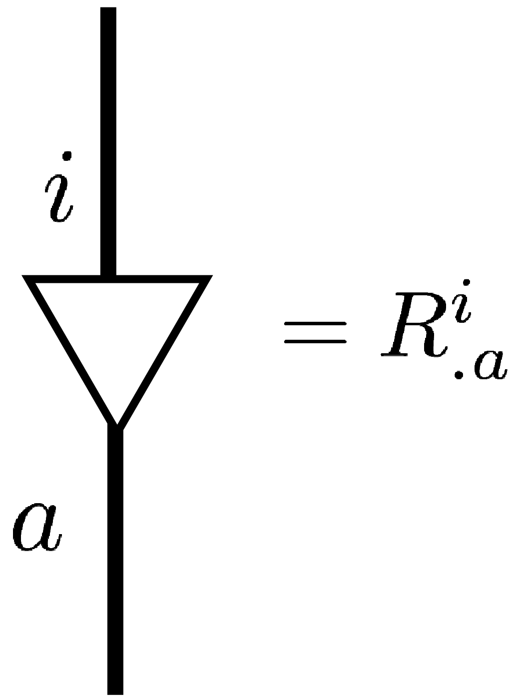


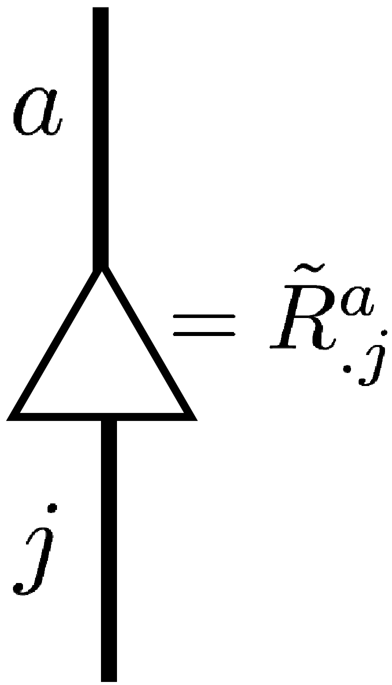
$$= \frac{1}{\sqrt{3!}} \epsilon_{ijk}$$

$$= \frac{1}{\sqrt{3!}} \epsilon^{ijk}$$

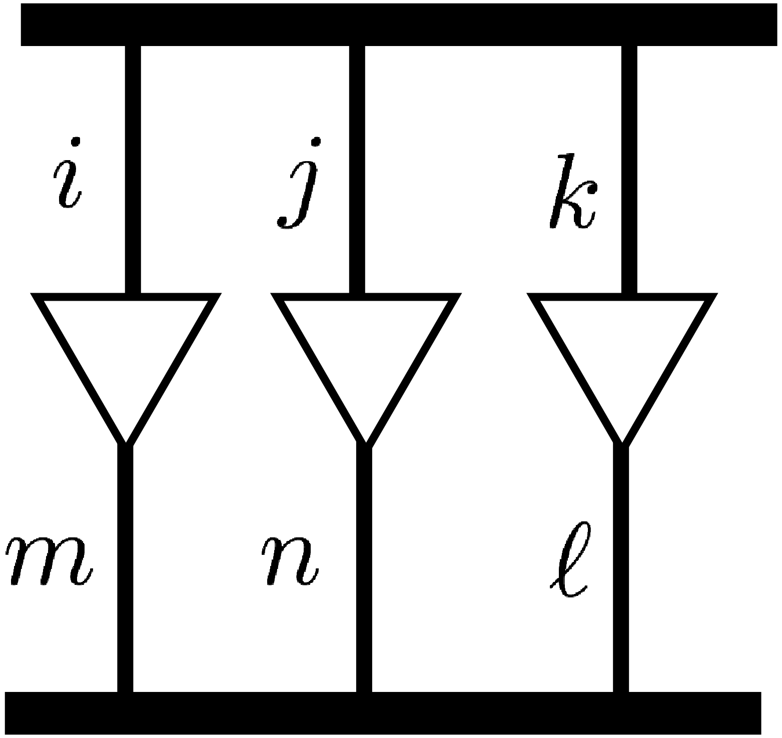
$$R\tilde{R} = 1, \qquad R^{i}_{\cdot a}\tilde{R}^{a}_{\cdot j} = \delta^{i}_{j}$$





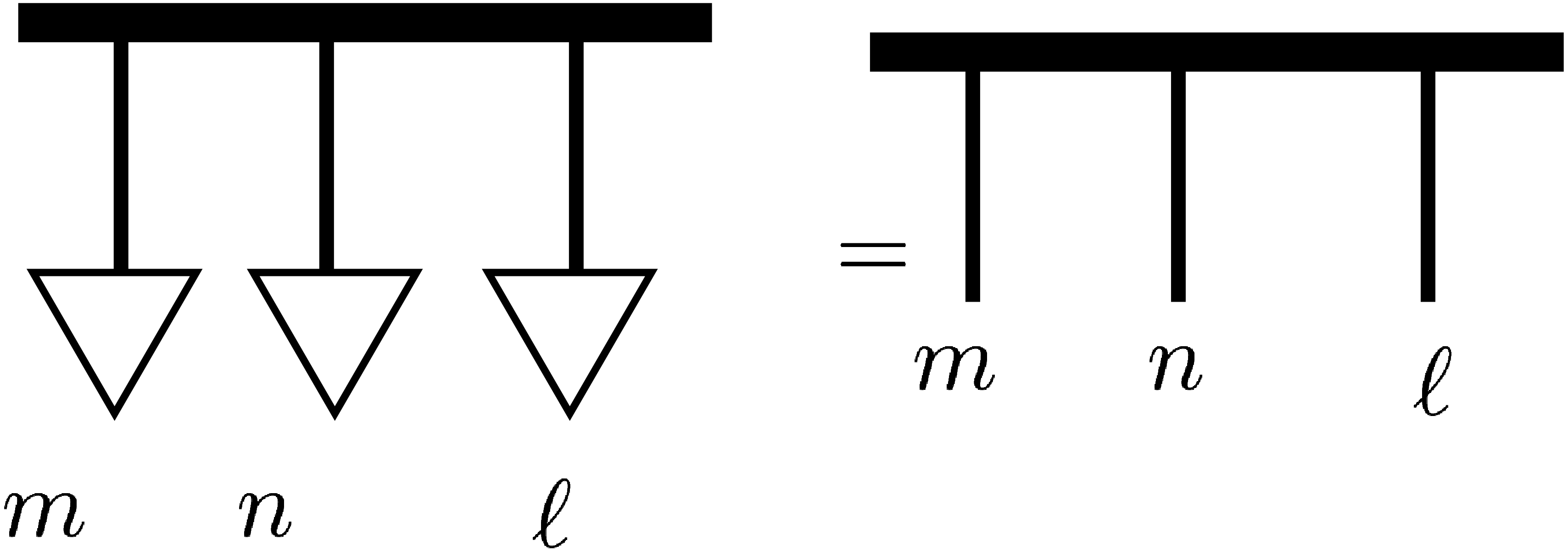


$$\det R = \frac{1}{3!}\epsilon_{ijk}\epsilon^{mn\ell} R^{i}_{.m} R^{j}_{.n} R^{k}_{.\ell} = 1$$

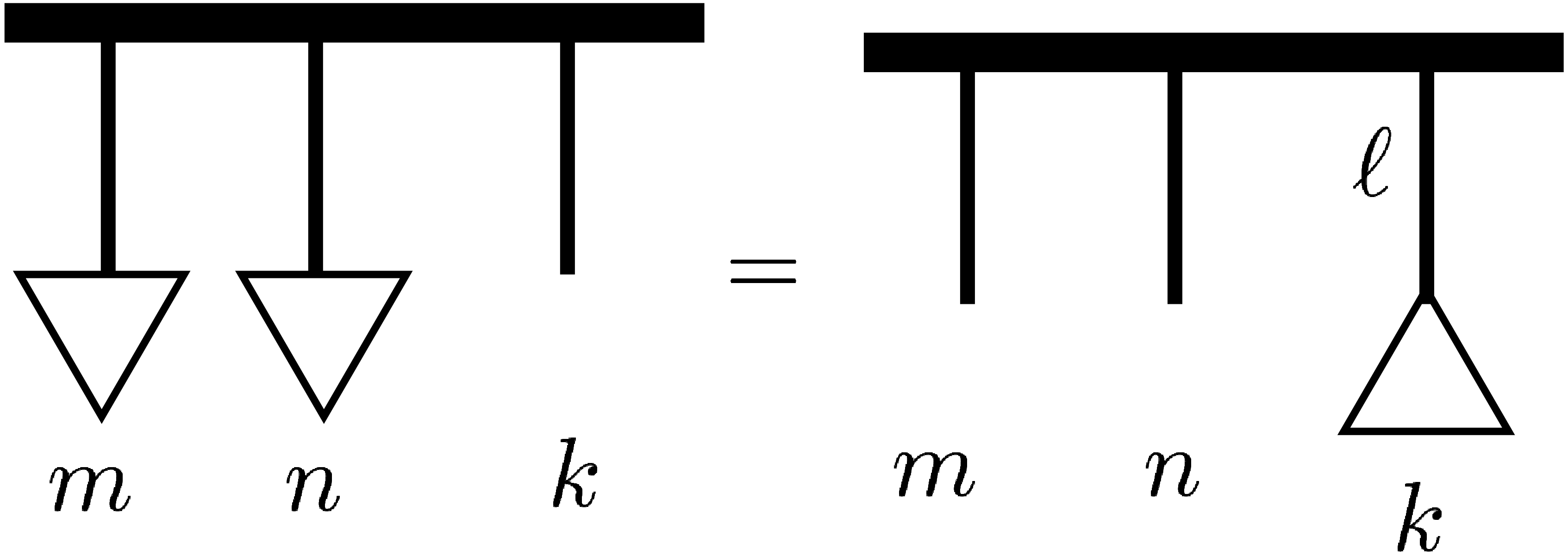


$$= 1$$

$$\epsilon_{ijk} R^{i}_{.m} R^{j}_{.n} R^{k}_{.\ell} = \epsilon_{mn\ell}$$



$$\epsilon_{ijk} R^{i}_{.m} R^{j}_{.n} = \epsilon_{mn\ell} R^{.\ell}_{k} = \epsilon_{mn\ell} \tilde{R}^{\ell}_{.k}$$



Ref: Landau & Lifshitz, V2

Now consider four-tensors. The space components (i, k, = 1, 2, 3) of the antisymmetric tensor A^{ik} form a three-dimensional antisymmetric tensor with respect to purely spatial transformations; according to our statement its components can be expressed in terms of the components of a three-dimensional axial vector. With respect to these same transformations the components A^{01}, A^{02}, A^{03} form a three-dimensional polar vector. Thus the components of an antisymmetric four-tensor can be written as a matrix:

$$(A^{ik}) = \begin{vmatrix} 0 & p_x & p_y & p_z \\ -p_x & 0 & -a_z & a_y \\ -p_y & a_z & 0 & -a_x \\ -p_z & -a_y & a_x & 0 \end{vmatrix}, \tag{6.10}$$

where, with respect to spatial transformations, **p** and **a** are polar and axial vectors, respectively.